

An $\mathfrak{su}(1|1)$ -Invariant S-Matrix with Dynamic Representations

NIKLAS BEISERT

*Joseph Henry Laboratories
Princeton University
Princeton, NJ 08544, USA*

nbeisert@princeton.edu

Abstract

The spin chains originating from large- N conformal gauge theories are of a special kind: The Hamiltonian is *not invariant* under the symmetry algebra, it is rather *a part* of it. This leads to interesting properties within the asymptotic Bethe ansatz. Here we study an S-matrix with $\mathfrak{u}(1|1)$ symmetry which arises in a long-range spin chain with fundamental spins of $\mathfrak{su}(2|1)$.

1 Introduction

Field theories with extended supersymmetries have turned out to be a reliable source for unexpected further symmetries. The most famous examples are probably the maximally extended supergravities [1] with exceptional groups as hidden symmetries [2]. Similarly, maximally supersymmetric $\mathcal{N} = 4$ gauge theory in four dimensions [3] has been known for a long time to be one of the few finite quantum field theories [4]. Due to masslessness, the quantum theory is superconformal and has the global symmetry algebra $\mathfrak{psu}(2, 2|4)$. In recent years it has become clear that there is even more symmetry when restricting to the large- N limit: *Integrability*. In the study of the AdS/CFT correspondence [5], Minahan and Zarembo found an integrable structure [6] which was subsequently extended to higher perturbative orders [7, 8] and all local operators [9].¹ Due to the apparent

¹Lipatov's remarks on integrability in $\mathcal{N} = 4$ SYM [10] were well ahead of these developments.

integrability, planar anomalous dimensions can now be computed by means of Bethe ansätze [6, 7, 11–13]. The latter have proved to be extremely useful for studies of the AdS/CFT correspondence [14]. For reviews of the subject of integrability in $\mathcal{N} = 4$ SYM, please refer to [15].

In [13] an all-loop solution was proposed for spectrum of the $\mathfrak{su}(1|2)$ sector of $\mathcal{N} = 4$ SYM. This solution involved an interesting S-matrix obeying the Yang-Baxter relation. The S-matrix was ‘extrapolated’ from its first few perturbative orders which were derived from the underlying Hamiltonian. It has a remarkably simple form, but its nature mostly remained obscure. It is one of the purposes of this note to clarify the origin of the S-matrix from a representation theory point of view. For instance, in most known cases the S-matrix is uniquely determined by the symmetries it obeys and the representations of the scattering objects. This will turn out to be the case here as well.

2 The $\mathfrak{su}(1|2)$ Sector of $\mathcal{N} = 4$ SYM

The $\mathfrak{su}(1|2)$ sector of $\mathcal{N} = 4$ SYM is equivalent to a long-range spin chain where each spin takes one out of three orientations: The bosonic orientation \mathcal{Z} is considered to be the vacuum while the other two orientations ϕ and ψ , which are bosonic and fermionic, respectively, are considered to be the excitations of the vacuum. The Hamiltonian of this spin chain is of the long-range type as introduced in [7]. In other words, the Hamiltonian

$$\mathcal{H}(\lambda) = \mathcal{H}_0 + \lambda \mathcal{H}_1 + \lambda^2 \mathcal{H}_2 + \dots \quad (1)$$

is a perturbative deformation of a nearest-neighbour spin interaction \mathcal{H}_0 with the higher orders \mathcal{H}_r in the small coupling λ acting among $r+2$ adjacent spins. For the given sector the Hamiltonian was computed up to second order in [8].

3 The Asymptotic Bethe Ansatz

To construct the integrable structure of such a long-range system remains a difficult task. The R-matrix suits this purpose well only for common nearest-neighbour spin chains; it is not yet clear if and how it might be applied here. Nevertheless, these difficulties can be overcome by considering asymptotic states and the S-matrix of the excitations [16, 12]. In an asymptotic state all excitations are sufficiently separated along the spin chain, e.g.

$$|\phi_1 \dots \psi_K\rangle = \sum_{a_1 \ll \dots \ll a_K} e^{ia_1 p_1 + \dots + ia_K p_K} |\dots \mathcal{Z} \mathcal{Z} \dots \overset{a_1}{\downarrow} \phi \dots \overset{\dots}{\downarrow} \dots \overset{a_K}{\downarrow} \psi \dots \mathcal{Z} \mathcal{Z} \dots\rangle. \quad (2)$$

Here, the indices $k = 1, \dots, K$ refer to the momenta p_k of the excitation. The crucial insight is that the Hamiltonian is *homogeneous* and *local*. These asymptotic states are therefore asymptotically eigenstates of the Hamiltonian. Homogeneity leads to the plane wave factors in (2) and locality makes the propagation of the individual excitations independent of each other. The only violation of the eigenstate equation for the state

in (2) comes from the vicinity of the boundaries of the asymptotic region; when two excitations come too close they will interact non-trivially.

In order to construct exact eigenstates, the various asymptotic regions have to be stitched up in a suitable way. For an integrable system, this can be achieved by a pairwise S-matrix. For instance, to combine the two asymptotic regions where the first two excitations are in either ordering, one would use

$$(1 + \mathcal{S}_{12}) |\phi_1 \phi_2 \dots\rangle = |\phi_1 \phi_2 \dots\rangle + S_{12} |\phi_2 \phi_1 \dots\rangle. \quad (3)$$

Here S_{12} represents some phase factor due to the application of the S-matrix. A generic asymptotic eigenstate² can be constructed as follows

$$|\Psi\rangle = \sum_{\pi \in S_K} \mathcal{S}_\pi |\phi_1 \dots \psi_K\rangle, \quad (4)$$

where \mathcal{S}_π is a product of nearest-neighbour S-matrices which permute according to the permutation π .

4 The $\mathfrak{su}(1|1)$ Asymptotic Symmetry Algebra

The Hamiltonian is part of the symmetry algebra $\mathfrak{su}(2|1)$ which acts on the spin chain. The residual algebra which leaves the number of excitations invariant is $\mathfrak{u}(1|1)$. It consists of the two supercharges $\mathfrak{Q}, \mathfrak{S}$, the outer automorphism \mathfrak{B} and the central charge \mathfrak{C} . The central charge contains the Hamiltonian

$$\mathfrak{C}(\lambda) = \mathfrak{C}_0 + \lambda \mathcal{H}(\lambda). \quad (5)$$

The non-trivial commutators of the $\mathfrak{u}(1|1)$ algebra are given by

$$\{\mathfrak{Q}, \mathfrak{S}\} = \mathfrak{C}, \quad [\mathfrak{B}, \mathfrak{Q}] = -2\mathfrak{Q}, \quad [\mathfrak{B}, \mathfrak{S}] = +2\mathfrak{S}, \quad (6)$$

The algebra also has an invariant quadratic Casimir operator \mathfrak{J}^2 , it reads

$$\mathfrak{J}^2 = 2[\mathfrak{Q}, \mathfrak{S}] + \{\mathfrak{B}, \mathfrak{C}\}. \quad (7)$$

We can now construct a representation on a single excitation. The most general solution of the algebra relations is given by

$$\begin{aligned} \mathfrak{B} |\phi\rangle &= (b+1) |\phi\rangle, & \mathfrak{B} |\psi\rangle &= (b-1) |\psi\rangle, \\ \mathfrak{Q} |\phi\rangle &= q |\psi\rangle, & \mathfrak{Q} |\psi\rangle &= 0, \\ \mathfrak{S} |\phi\rangle &= 0, & \mathfrak{S} |\psi\rangle &= c/q |\phi\rangle, \\ \mathfrak{C} |\phi\rangle &= c |\phi\rangle, & \mathfrak{C} |\psi\rangle &= c |\psi\rangle, \\ \mathfrak{J}^2 |\phi\rangle &= 2bc |\phi\rangle, & \mathfrak{J}^2 |\psi\rangle &= 2bc |\phi\rangle. \end{aligned} \quad (8)$$

²An asymptotic eigenstate is defined to be a state which can be completed by non-asymptotic contributions (nearby excitations) to obtain an exact eigenstate.

The coefficient q is an unphysical quantity which reflects the difference of normalizations between bosonic and fermionic excitations. Note that the representation agrees with the expansion of generators derived in [8] when applied to single excitations restricted to the $\mathfrak{su}(1|2)$ sector. There, c represents the energy and q contains some of the unphysical constants related to similarity transformations.

Let us denote the above representation (8) with central charge c and hypercharge b by $(\mathbf{1}|\mathbf{1})_{c,b}$. The representation on asymptotic states with K excitations is the tensor product $(\mathbf{1}|\mathbf{1})_{c_1,b_1} \otimes (\mathbf{1}|\mathbf{1})_{c_2,b_2} \otimes \dots \otimes (\mathbf{1}|\mathbf{1})_{c_K,b_K}$.

5 The Invariant S-Matrix

The S-matrix is an invariant operator acting on two modules by interchanging them

$$\mathcal{S}_{12} : (\mathbf{1}|\mathbf{1})_{c_1} \otimes (\mathbf{1}|\mathbf{1})_{c_2} \rightarrow (\mathbf{1}|\mathbf{1})_{c_2} \otimes (\mathbf{1}|\mathbf{1})_{c_1}. \quad (9)$$

We shall write it as a product of an operator \mathcal{R}_{12} (R-matrix) and the (graded) permutation \mathcal{P}_{12}

$$\mathcal{S}_{12} = \mathcal{P}_{12} \mathcal{R}_{12}(a_1, a_2), \quad \mathcal{R}_{12} : (\mathbf{1}|\mathbf{1})_{c_1} \otimes (\mathbf{1}|\mathbf{1})_{c_2} \rightarrow (\mathbf{1}|\mathbf{1})_{c_1} \otimes (\mathbf{1}|\mathbf{1})_{c_2}. \quad (10)$$

The R-matrix depends on two spectral parameters $a_k = a(p_k)$, which are themselves function of the particle momenta. The permutation is clearly $\mathfrak{u}(1|1)$ invariant and the same must therefore be true for the R-matrix. The latter can therefore be written as a sum over projectors to irreducible components. The tensor product in question decomposes into two similar irreducible modules $(\mathbf{1}|\mathbf{1})_{c_1+c_2, b_1+b_2+1}$ and $(\mathbf{1}|\mathbf{1})_{c_1+c_2, b_1+b_2-1}$. These two modules can be distinguished by the quadratic Casimir on the tensor product

$$\mathfrak{J}_{12}^2 = \mathfrak{J}_1^2 + 2\mathfrak{J}_1 \cdot \mathfrak{J}_2 + \mathfrak{J}_2^2 = 2b_1c_1 + 2b_2c_2 + 2\mathfrak{B}_1\mathfrak{C}_2 + 2\mathfrak{C}_1\mathfrak{B}_2 + 4\mathfrak{Q}_1\mathfrak{S}_2 - 4\mathfrak{S}_1\mathfrak{Q}_2. \quad (11)$$

Because there are only two irreps in the tensor product, all invariant operators can now be written as a linear combination of the identity operator \mathcal{I}_{12} and the quadratic Casimir \mathfrak{J}_{12}^2 . In particular this applies to the square of \mathfrak{J}_{12}^2

$$(\mathfrak{J}_{12}^2)^2 = 4(b_1 + b_2)(c_1 + c_2) \mathfrak{J}_{12}^2 - 4(b_1 + b_2 + 1)(b_1 + b_2 - 1)(c_1 + c_2)^2 \mathcal{I}_{12}. \quad (12)$$

The same applies to the R-matrix which we can write as

$$\mathcal{R}_{12}(a_1, a_2) = R_{12,1}(a_1, a_2) \mathcal{I}_{12} + R_{12,2}(a_1, a_2) \mathfrak{J}_{12}^2 \quad (13)$$

with some coefficients $R_{12,1}, R_{12,2}$ to be determined. It should obey the unitarity and Yang-Baxter relations

$$\mathcal{R}_{12}\mathcal{R}_{21} = \mathcal{I}_{12}, \quad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (14)$$

The standard solution for the coefficients in the R-matrix is

$$\begin{aligned} R_{12,1}(a_1, a_2) &= \frac{a_2 - a_1 - \frac{i}{2}(b_1 + b_2)(c_1 + c_2)}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} R_{12,0}(a_1, a_2), \\ R_{12,2}(a_1, a_2) &= \frac{\frac{i}{4}}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} R_{12,0}(a_1, a_2) \end{aligned} \quad (15)$$

with some undetermined phase $R_{12,0}(a_1, a_2)$ obeying

$$R_{12,0}(a_1, a_2)R_{12,0}(a_2, a_1) = 1. \quad (16)$$

Let us for convenience drop this overall phase, $R_{12,0} = 1$. We apply the R-matrix to a two-particle state and find

$$\begin{aligned} \mathcal{R}_{12}(a_1, a_2) |\phi_1 \phi_2\rangle &= \frac{a_2 - a_1 + \frac{i}{2}(c_1 + c_2)}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} |\phi_1 \phi_2\rangle, \\ \mathcal{R}_{12}(a_1, a_2) |\phi_1 \psi_2\rangle &= \frac{a_2 - a_1 + \frac{i}{2}(c_2 - c_1)}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} |\phi_1 \psi_2\rangle + \frac{ic_2}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} \frac{q_1}{q_2} |\psi_1 \phi_2\rangle, \\ \mathcal{R}_{12}(a_1, a_2) |\psi_1 \phi_2\rangle &= \frac{a_2 - a_1 + \frac{i}{2}(c_1 - c_2)}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} |\psi_1 \phi_2\rangle + \frac{ic_1}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} \frac{q_2}{q_1} |\phi_1 \psi_2\rangle, \\ \mathcal{R}_{12}(a_1, a_2) |\psi_1 \psi_2\rangle &= \frac{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)}{a_2 - a_1 - \frac{i}{2}(c_1 + c_2)} |\psi_1 \psi_2\rangle. \end{aligned} \quad (17)$$

More general results for quantum deformed symmetry algebras can be found in [17, 18]. Note that above R-matrix coincides with the results in [18] when setting $\alpha, \beta = c_1, c_2$, $q = \exp(\kappa)$, $x = \exp(2i\kappa(a_2 - a_1))$ and sending the deformation parameter $\kappa \rightarrow 0$.

6 The S-Matrix for the $\mathfrak{su}(1|2)$ Sector

As the next step, we apply the above results to asymptotic states of the spin chain for $\mathcal{N} = 4$ SYM. There, the spectral parameter a of a particle is given by a function of the momentum, $a = a(p)$. Furthermore, the central charge is interpreted as the energy of a state as $c = 1 + \lambda e$ which itself is given through the dispersion relation $e = e(p)$, i.e. we have $c = c(p)$. We can now perform a change of parameters as follows

$$a = \frac{1}{2}(x^+ + x^-), \quad c = -i(x^+ - x^-), \quad (18)$$

where now x^\pm are given as some functions of the momentum, $x^\pm = x^\pm(p)$. Substituting this in the S-matrix, we obtain

$$\begin{aligned} \mathcal{S}_{12} |\phi_1 \phi_2\rangle &= \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} |\phi_2 \phi_1\rangle, \\ \mathcal{S}_{12} |\phi_1 \psi_2\rangle &= \frac{x_2^+ - x_1^+}{x_2^- - x_1^+} |\psi_2 \phi_1\rangle + \frac{x_2^+ - x_2^-}{x_2^- - x_1^+} \frac{q_1}{q_2} |\phi_2 \psi_1\rangle, \\ \mathcal{S}_{12} |\psi_1 \phi_2\rangle &= \frac{x_2^- - x_1^-}{x_2^- - x_1^+} |\phi_2 \psi_1\rangle + \frac{x_1^+ - x_1^-}{x_2^- - x_1^+} \frac{q_2}{q_1} |\psi_2 \phi_1\rangle, \\ \mathcal{S}_{12} |\psi_1 \psi_2\rangle &= -\frac{x_2^- - x_1^+}{x_2^- - x_1^+} |\psi_2 \psi_1\rangle. \end{aligned} \quad (19)$$

This is precisely the S-matrix found in [13] for the $\mathfrak{su}(1|2)$ sector of $\mathcal{N} = 4$ SYM. There, the functions $x^\pm(u)$ are defined intrinsically through the equations

$$\frac{x^+}{x^-} = \exp(ip), \quad x^+ + \frac{\lambda}{x^+} - x^- - \frac{\lambda}{x^-} = i. \quad (20)$$

In particular, this means that the energy e of an excitation, which is related to its central charge via $c = 1 + \lambda e$, is given by

$$e = \frac{i}{x^+} - \frac{i}{x^-}. \quad (21)$$

This agrees with the dispersion relation used in [13].

7 XXZ Spin Chain

In conventional nearest-neighbour spin chains the representation of all excitations is the same. In particular, it does not depend on the momentum and therefore one might set $c = 1$ for all excitations. In that case

$$x^\pm = a \pm \frac{i}{2} \quad (22)$$

and the S-matrix (19) reduces to that of the standard $\mathfrak{su}(2|1)$ spin chain with nearest-neighbour interactions (we can also set $q_1 = q_2$).

Inspired by the simplicity of (19) one might try to employ the notation using x^\pm also to other well-known cases. Let us investigate the XXZ spin chain which is a deformation of the $\mathfrak{su}(2)$ Heisenberg chain with two spin orientations labelled by $|\mathcal{Z}\rangle$ and $|\phi\rangle$. The Hamiltonian reads

$$\mathcal{H}_{12} = \frac{1}{4}(r^{+1} + r^{-1})(\mathcal{I}_1\mathcal{I}_2 - \sigma_1^z\sigma_2^z) - \frac{1}{2}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y) \quad (23)$$

The anisotropy parameter is $\frac{1}{2}(r^{+1} + r^{-1})$. Alternatively, we shall investigate a similar spin chain where we replace the bosonic state $|\phi\rangle$ by the fermionic $|\psi\rangle$. Using the above representation (8) with $c = q = 1$ and $b = 0$ the Hamiltonian reads

$$\mathcal{H}_{12} = \frac{1}{2}(r^{+1} + r^{-1})(\mathcal{I}_1\mathcal{I}_2 - \frac{1}{2}\{\mathfrak{B}_1, \mathfrak{C}_2\}) - [\mathfrak{Q}_1, \mathfrak{S}_2]. \quad (24)$$

The nearest-neighbour Hamiltonians \mathcal{H}_{12} act on a pair of spins as³

$$\begin{aligned} \mathcal{H}_{12}|\mathcal{Z}\mathcal{Z}\rangle &= 0, & \mathcal{H}_{12}|\mathcal{Z}\mathcal{Z}\rangle &= 0, \\ \mathcal{H}_{12}|\mathcal{Z}\phi\rangle &= r^{-1}|\mathcal{Z}\phi\rangle - |\phi\mathcal{Z}\rangle, & \mathcal{H}_{12}|\mathcal{Z}\psi\rangle &= r^{-1}|\mathcal{Z}\psi\rangle - |\psi\mathcal{Z}\rangle, \\ \mathcal{H}_{12}|\phi\mathcal{Z}\rangle &= r^{+1}|\phi\mathcal{Z}\rangle - |\mathcal{Z}\phi\rangle, & \mathcal{H}_{12}|\psi\mathcal{Z}\rangle &= r^{+1}|\psi\mathcal{Z}\rangle - |\mathcal{Z}\psi\rangle, \\ \mathcal{H}_{12}|\phi\phi\rangle &= 0, & \mathcal{H}_{12}|\psi\psi\rangle &= (r^{+1} + r^{-1})|\psi\psi\rangle. \end{aligned} \quad (25)$$

It is straightforward to perform the coordinate space Bethe ansatz for these systems. We can now define the x^\pm as

$$e^{ip} = \frac{x^+}{rx^-}, \quad r^{-1}x^+ - r^{+1}x^- = \frac{i}{2}(r^{+1} + r^{-1}). \quad (26)$$

The second equation permits us to solve both x^\pm in terms of one spectral parameter u

$$x^\pm = r^{\pm 1}u \pm \frac{i}{2}. \quad (27)$$

³The interaction $|\mathcal{Z}\phi\rangle \rightarrow +|\mathcal{Z}\phi\rangle, |\phi\mathcal{Z}\rangle \rightarrow -|\phi\mathcal{Z}\rangle$ is a boundary contribution and can be dropped.

The dispersion relation is then similar to (21)

$$e = \frac{1}{2} (r^{+1} + r^{-1}) \left(\frac{i}{x^+} - \frac{i}{x^-} \right) \quad (28)$$

and the S-matrix (a factor) is simply

$$\mathcal{S}_{12} |\phi_1 \phi_2\rangle = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} |\phi_2 \phi_1\rangle, \quad \mathcal{S}_{12} |\psi_1 \psi_2\rangle = -|\psi_2 \psi_1\rangle. \quad (29)$$

for bosons and fermions, respectively. Consequently, the Bethe equations read

$$\left(\frac{rx_k^-}{x_k^+} \right)^L \prod_{\substack{j=1 \\ j \neq k}}^K \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} = 1, \quad \left(\frac{rx_k^-}{x_k^+} \right)^L = 1. \quad (30)$$

Let us relate our parametrisation to the common one involving trigonometric functions, cf. [19] and the review [20]. The variables x^\pm are then related to a spectral parameter ν as follows

$$x^\pm = \coth(\kappa) \sin\left(\kappa\left(\nu \pm \frac{i}{2}\right)\right) \exp\left(-i\kappa\left(\nu \pm \frac{i}{2}\right)\right), \quad r = \exp(\kappa). \quad (31)$$

Furthermore, the parameter r has been replaced by κ . The propagation phase in the trigonometric form reads

$$e^{ip} = \frac{x^+}{rx^-} = \frac{\sin\left(\kappa\left(\nu + \frac{i}{2}\right)\right)}{\sin\left(\kappa\left(\nu - \frac{i}{2}\right)\right)} \quad (32)$$

and the scattering term becomes a function of the difference of spectral parameters

$$\frac{x_1^+ - x_2^-}{x_1^- - x_2^+} = \frac{\sin\left(\kappa(\nu_1 - \nu_2 + i)\right)}{\sin\left(\kappa(\nu_1 - \nu_2 - i)\right)}. \quad (33)$$

The dependence on $\nu_1 - \nu_2$ is the benefit of the trigonometric parametrisation; if this is not desired, one can employ the algebraic formulation in (29), cf. also [21] for a similar parametrisation.

8 Quantum Algebra Deformed Spin Chains

The algebraic parametrisation of XXZ-like spin chains also generalises to symmetry algebras of higher rank. Here we consider deformations of chains with spins transforming the fundamental representation of $\mathfrak{su}(3)$ and $\mathfrak{su}(2|1)$. The generic model with these properties has recently been investigated in [22]. The three states of a spin are $|\mathcal{Z}\rangle, |\phi\rangle, |\psi\rangle$, the first two are always bosonic and the statistics of the third depends on the algebra. The Hamiltonian acts on two different spins as follows [22]

$$\begin{aligned} \mathcal{H}_{12} |\mathcal{Z}\phi\rangle &= r^{-1} |\mathcal{Z}\phi\rangle - |\phi\mathcal{Z}\rangle, & \mathcal{H}_{12} |\phi\mathcal{Z}\rangle &= r^{+1} |\phi\mathcal{Z}\rangle - |\mathcal{Z}\phi\rangle, \\ \mathcal{H}_{12} |\mathcal{Z}\psi\rangle &= r^{-1} |\mathcal{Z}\psi\rangle - |\psi\mathcal{Z}\rangle, & \mathcal{H}_{12} |\psi\mathcal{Z}\rangle &= r^{+1} |\psi\mathcal{Z}\rangle - |\mathcal{Z}\psi\rangle, \\ \mathcal{H}_{12} |\phi\psi\rangle &= r^{-1} |\phi\psi\rangle - |\psi\phi\rangle, & \mathcal{H}_{12} |\psi\phi\rangle &= r^{+1} |\psi\phi\rangle - |\phi\psi\rangle. \end{aligned} \quad (34)$$

For two equal spins we assume for $\mathfrak{su}(3)$

$$\mathcal{H}_{12}|\mathcal{Z}\mathcal{Z}\rangle = 0, \quad \mathcal{H}_{12}|\phi\phi\rangle = 0, \quad \mathcal{H}_{12}|\psi\psi\rangle = 0 \quad (35)$$

and for $\mathfrak{su}(2|1)$

$$\mathcal{H}_{12}|\mathcal{Z}\mathcal{Z}\rangle = 0, \quad \mathcal{H}_{12}|\phi\phi\rangle = 0, \quad \mathcal{H}_{12}|\psi\psi\rangle = (r^{+1} + r^{-1})|\psi\psi\rangle. \quad (36)$$

In the coordinate space Bethe ansatz we can use the above definition of parameters x^\pm (26,27) and obtain the same dispersion relation (28) for both excitations ϕ, ψ above the vacuum of \mathcal{Z} 's, cf. [22]. Even more, the S-matrix also takes almost the above form (19). The scattering of ϕ and ψ merely picks up factors of r

$$\begin{aligned} \mathcal{S}_{12}|\phi_1\psi_2\rangle &= r^{-1} \frac{x_2^+ - x_1^+}{x_2^- - x_1^+} |\psi_2\phi_1\rangle + \frac{x_2^+ - x_2^-}{x_2^- - x_1^+} \frac{q_1}{q_2} |\phi_2\psi_1\rangle, \\ \mathcal{S}_{12}|\psi_1\phi_2\rangle &= r^{+1} \frac{x_2^- - x_1^-}{x_2^- - x_1^+} |\phi_2\psi_1\rangle + \frac{x_1^+ - x_1^-}{x_2^- - x_1^+} \frac{q_2}{q_1} |\psi_2\phi_1\rangle. \end{aligned} \quad (37)$$

The scattering of identical particles for $\mathfrak{su}(3)$ is

$$\mathcal{S}_{12}|\phi_1\phi_2\rangle = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} |\phi_2\phi_1\rangle, \quad \mathcal{S}_{12}|\psi_1\psi_2\rangle = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} |\psi_2\psi_1\rangle, \quad (38)$$

whereas for $\mathfrak{su}(2|1)$ we get the same as in (19)

$$\mathcal{S}_{12}|\phi_1\phi_2\rangle = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} |\phi_2\phi_1\rangle, \quad \mathcal{S}_{12}|\psi_1\psi_2\rangle = -\frac{x_2^- - x_1^+}{x_2^- - x_1^+} |\psi_2\psi_1\rangle. \quad (39)$$

Note, however, that this S-matrix satisfies the Yang-Baxter equation only if the relation (26) for x^+ and x^- is fulfilled. This is in contrast with the S-matrix (19) which satisfies the YBE even if x^+ and x^- are independent.

The above S-matrix for deformed $\mathfrak{su}(2|1)$ agrees with the one obtained in [18] when setting $\alpha = \beta = 1$ and identifying

$$r = q, \quad x^\pm = \frac{i}{2} \frac{q^{+1} + q^{-1}}{q^{+1} - q^{-1}} (q^{\pm 1} x^{-1} - 1). \quad (40)$$

The nested Bethe ansatz for this system leads to a spectral parameter y^\pm for the auxiliary Bethe roots

$$y^\pm = r^{\pm 1} (v \pm \frac{i}{2}). \quad (41)$$

For the main Bethe equation we obtain

$$\left(\frac{rx_k^-}{x_k^+} \right)^L \prod_{\substack{j=1 \\ j \neq k}}^K \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \prod_{j=1}^J r^{+1} \frac{x_k^- - v_j}{x_k^+ - v_j} = 1. \quad (42)$$

For $\mathfrak{su}(3)$ and $\mathfrak{su}(2|1)$, the auxiliary equations read, respectively

$$\prod_{\substack{j=1 \\ j \neq k}}^J \frac{y_k^+ - y_j^-}{y_k^- - y_j^+} \prod_{j=1}^K r^{+1} \frac{y_k^- - u_j}{y_k^+ - u_j} = 1, \quad \prod_{j=1}^K r^{+1} \frac{y_k^- - u_j}{y_k^+ - u_j} = 1. \quad (43)$$

9 Conclusions

In this note we have studied the S-matrix and asymptotic Bethe equations for a particular long-range spin chain with $\mathfrak{su}(2|1)$ symmetry which arises in a sector of planar $\mathcal{N} = 4$ super Yang-Mills theory. The model has the interesting feature that the representation of an excitation depends on its momentum along the spin chain. This property is reflected by the S-matrix which takes an interesting generic form using the spectral parameters x^\pm . We have also considered nearest-neighbour chains with quantum deformed $\mathfrak{su}(3)$ and $\mathfrak{su}(2|1)$ symmetries. Their S-matrices and Bethe ansätze are usually formulated using trigonometric functions, but an alternative formulation using x^\pm -parameters leads to very similar structures as for the above $\mathfrak{su}(2|1)$ long-range chain.

We hope these results may improve the understanding of generic long-range spin chain models [23] and their relation to the well-known nearest-neighbour chains. The results may also be useful for the study of quantum strings on $AdS_5 \times S^5$ by means of Bethe ansätze [24], spin chains [25] and supersymmetric subsectors [26]. Finally, the findings can be generalized to the complete $\mathcal{N} = 4$ SYM spin chain model [27].

Acknowledgements. I thank Andreas Klümper, Charlotte Kristjansen, Nicolai Reshetikhin, Matthias Staudacher and Yao-Zhong Zhang for discussions and pointing out references. This work is supported in part by the U.S. National Science Foundation Grant No. PHY02-43680. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

References

- [1] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. B76, 409 (1978).
- [2] E. Cremmer and B. Julia, Phys. Lett. B80, 48 (1978).
E. Cremmer and B. Julia, Nucl. Phys. B159, 141 (1979).
B. de Wit and H. Nicolai, Class. Quant. Grav. 18, 3095 (2001), [hep-th/0011239](#).
- [3] F. Gliozzi, J. Scherk and D. I. Olive, Nucl. Phys. B122, 253 (1977).
L. Brink, J. H. Schwarz and J. Scherk, Nucl. Phys. B121, 77 (1977).
- [4] M. F. Sohnius and P. C. West, Phys. Lett. B100, 245 (1981).
P. S. Howe, K. S. Stelle and P. K. Townsend, Nucl. Phys. B236, 125 (1984).
L. Brink, O. Lindgren and B. E. W. Nilsson, Nucl. Phys. B212, 401 (1983).
- [5] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998), [hep-th/9711200](#).
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B428, 105 (1998), [hep-th/9802109](#).
E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998), [hep-th/9802150](#).
- [6] J. A. Minahan and K. Zarembo, JHEP 0303, 013 (2003), [hep-th/0212208](#).
- [7] N. Beisert, C. Kristjansen and M. Staudacher, Nucl. Phys. B664, 131 (2003), [hep-th/0303060](#).
- [8] N. Beisert, Nucl. Phys. B682, 487 (2004), [hep-th/0310252](#).
- [9] N. Beisert and M. Staudacher, Nucl. Phys. B670, 439 (2003), [hep-th/0307042](#).
- [10] L. N. Lipatov, in: *“Perspectives in Hadronic Physics”*, ed.: S. Boffi, C. Ciofi Degli Atti and M. Giannini, World Scientific (1998), Singapore.
L. N. Lipatov, Nucl. Phys. B548, 328 (1999), [hep-ph/9812336](#).
- [11] D. Serban and M. Staudacher, JHEP 0406, 001 (2004), [hep-th/0401057](#).
N. Beisert, V. Dippel and M. Staudacher, JHEP 0407, 075 (2004), [hep-th/0405001](#).
- [12] M. Staudacher, JHEP 0505, 054 (2005), [hep-th/0412188](#).

- [13] N. Beisert and M. Staudacher, Nucl. Phys. B727, 1 (2005), [hep-th/0504190](#).
- [14] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, JHEP 0309, 010 (2003), [hep-th/0306139](#).
N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, JHEP 0310, 037 (2003), [hep-th/0308117](#).
- [15] N. Beisert, Phys. Rept. 405, 1 (2005), [hep-th/0407277](#).
N. Beisert, Comptes Rendus Physique 5, 1039 (2004), [hep-th/0409147](#).
K. Zarembo, Comptes Rendus Physique 5, 1081 (2004), [hep-th/0411191](#).
J. Plefka, [hep-th/0507136](#).
- [16] B. Sutherland, Rocky Mountain J. Math. 8, 413 (1978).
- [17] P. P. Kulish and N. Y. Reshetikhin, Lett. Math. Phys. 18, 143 (1989).
N. Reshetikhin, Lett. Math. Phys. 20, 331 (1990).
L. Dabrowski and L.-Y. Wang, Phys. Lett. B266, 51 (1991).
H. Hinrichsen and V. Rittenberg, Phys. Lett. B275, 350 (1992), [hep-th/9110074](#).
N. Y. Reshetikhin, in: “*Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology*”, ed.: T. Curtright and L. Mezincescu, World Scientific (1992), River Edge, NJ, USA.
- [18] A. J. Bracken, M. D. Gould, Y.-Z. Zhang and G. W. Delius, J. Phys. A27, 6551 (1994), [hep-th/9405138](#).
G. W. Delius, M. D. Gould, J. R. Links and Y.-Z. Zhang, Int. J. Mod. Phys. A10, 3259 (1995), [hep-th/9408006](#).
Y.-Z. Zhang and M. D. Gould, J. Math. Phys. 40, 5264 (1999), [math.qa/9809156](#).
- [19] M. Gaudin, Phys. Rev. A4, 386 (1971).
F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, J. Phys. A20, 6397 (1987).
E. K. Sklyanin, J. Phys. A21, 2375 (1988).
- [20] L. D. Faddeev, [hep-th/9605187](#), in: “*Relativistic gravitation and gravitational radiation*”, ed.: J.-A. Marck and J.-P. Lasota, Cambridge University Press (1997), Cambridge.
- [21] S. A. Frolov, R. Roiban and A. A. Tseytlin, JHEP 0507, 045 (2005), [hep-th/0503192](#).
- [22] L. Freyhult, C. Kristjansen and T. Månsson, [hep-th/0510221](#).
- [23] N. Beisert and T. Klose, [hep-th/0510124](#).
- [24] G. Arutyunov, S. Frolov and M. Staudacher, JHEP 0410, 016 (2004), [hep-th/0406256](#).
- [25] N. Beisert, Fortsch. Phys. 53, 852 (2005), [hep-th/0409054](#).
- [26] L. F. Alday, G. Arutyunov and S. Frolov, [hep-th/0508140](#).
G. Arutyunov and S. Frolov, [hep-th/0510208](#).
- [27] N. Beisert, [hep-th/0511082](#).